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Contents

Abstract. We consider Hardy spaces associated to the conjugated Beltrami equation on doubly connected planar domains. There are two main differences with previous studies ([4]).

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First, while the simple connectivity plays an important role in [4], the multiple connectivity of the domain leads to unexpected difficulties. In particular, we make strong use of a suitable parametrization of an analytic function in a ring by its real part on one part of the boundary and by its imaginary part on the other. Then, we allow the coefficient in the conjugated Beltrami equation to belong to $W^{1,q}$ for some $q \in (2, +\infty]$, while it was supposed to be Lipschitz in [4]. We define Hardy spaces associated with the conjugated Beltrami equation and solve the corresponding Dirichlet problem. The same problems for generalized analytic function are also solved.
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1 Introduction

1.1 Notations

Throughout the paper, let $r_0 \in (0, 1)$ and define $\mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$, $\mathbb{D}_{r_0} := r_0\mathbb{D}$ and $G_2 := \{z \in \mathbb{C};\ r_0 < |z| < 1\}$. For all $r > 0$, let $\mathbb{T}_r$ stand for the circle with center 0 and radius r.
We will make use of the operators

$$\partial := \frac{1}{2}\left(\partial_x - i\partial_y\right) \text{ and } \overline{\partial} := \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

Let $\Omega \subset \mathbb{C}$ be a bounded domain, $p \in [1, +\infty]$. We identify $\mathbb{R}^2$ with $\mathbb{C}$, writing $\xi = x + iy$ for $\xi \in \mathbb{C}$ with $x, y \in \mathbb{R}$, and denote interchangeably the (differential of) planar Lebesgue measure by

$$dm(\xi) = dx\, dy = (i/2) d\xi \wedge d\overline{\xi}\,,$$

where $d\xi = dx + idy$ and $d\overline{\xi} = dx - idy$. A measurable function $f : \Omega \to \mathbb{C}$ belongs to $L^p(\Omega)$ if and only if

$$\|f\|^p_{L^p(\Omega)} := \int_\Omega |f(z)|^p\, dm(z) < +\infty,$$

and to $L^\infty(\Omega)$ if and only if

$$\text{ess sup}_{\ z\in\Omega} |f(z)| < +\infty.$$

If $p \in [1, +\infty]$, say that $f \in W^{1,p}(\Omega)$ if and only if $f \in L^p(\Omega)$ and ∂f and $\overline{\partial} f$ belong to $L^p(\Omega)$, and set

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\partial f\|_{L^p(\Omega)} + \left\|\overline{\partial} f\right\|_{L^p(\Omega)}.$$

Finally, denote by $L^p_{\mathbb{R}}(\Omega)$ (resp. $W^{1,p}_{\mathbb{R}}(\Omega)$) the real subspace of $L^p(\Omega)$ (resp. $W^{1,p}(\Omega)$) made of real-valued functions.
Say that a sequence $\xi_n \in G_2$ approaches $\xi \in \partial G_2$ non tangentially if it converges to ξ while no limit point of $(\xi_n - \xi)/|\xi_n - \xi|$ belongs to the tangent line to ∂G_2 at ξ. A function f on G_2 has non tangential limit ℓ at ξ if $f(\xi_n)$ tends to ℓ for any sequence ξ_n which approaches ξ non tangentially.
If $A(f)$ and $B(f)$ are two quantities depending on a function f ranging in a set E, say that $A(f) \sim B(f)$ if and only if there exists $C > 0$ such that, for all $f \in E$,

$$C^{-1}A(f) \leq B(f) \leq CA(f).$$

1.2 The conjugated Beltrami equation

Let $\nu \in W^{1,\infty}_{\mathbb{R}}(G_2)$ with $\|\nu\|_\infty < 1$ and $p \in (1,+\infty)$. In [4], we focused on the Dirichlet problem for the *conjugated* Beltrami equation:

$$\overline{\partial} f = \nu \overline{\partial f} \text{ in } \mathbb{D}. \tag{1}$$

Given $\varphi \in L^p_{\mathbb{R}}(\mathbb{T}_1)$, we proved that there exists a solution f of (1) satisfying

$$\text{Re tr } f = \varphi \text{ on } \mathbb{T}_1, \tag{2}$$

with

$$\operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} < +\infty\,, \tag{3}$$

where

$$\|f\|_{L^p(\mathbb{T}_r)} := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{1/p}.$$

The fact that f solves (1) and satisfies (3) entails that f has a non tangential limit almost everywhere on $\mathbb{T}_1$, denoted by tr f, and the trace in (2) has to be understood in this sense. Moreover, f is unique up to a purely imaginary constant, and if we normalize f by

$$\int_0^{2\pi} \text{Im tr } f(e^{i\theta})d\theta = 0,$$

then f is unique and

$$\operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} \leq C_p \|\varphi\|_{L^p(\mathbb{T}_1)}\,.$$

The space of solutions of (1) satisfying (3) is a Hardy space on $\mathbb{D}$, denoted by $H^p_\nu(\mathbb{D})$, which shares many properties of the classical $H^p(\mathbb{D})$ space. Note that, when $\nu = 0$ in $\mathbb{D}$, (1) exactly means that f is holomorphic and the solution of the Dirichlet problem (2) belongs to the classical $H^p(\mathbb{D})$ space.

In the present work, we investigate the Dirichlet problem for the conjugated Beltrami equation in a **doubly connected** domain $\mathbb{D}_2$ with analytic boundary. For simplicity of the presentation, we will restrict ourselves to the case of the ring $G_2 = \{z \in \mathbb{C};\ r_0 < |z| < 1\}$. Since any $\mathbb{D}_2$ with analytic boundary is conformally equivalent to G_2 with a conformal map continuous up to the boundary, for some unique $r_0 \in (0,1)$ (see [12], see also [13]), all the results of Sections 2, 3 and 4 below remain valid in $\mathbb{D}_2$. An important difference with the case of simply connected domains, due to the fact that the boundary has now two connected components, is that, in the Dirichlet problem, we prescribe the real part of the solution on one part of the boundary and the imaginary part on the other. Another difference with [4] is that we only assume that $\nu \in W^{1,q}_{\mathbb{R}}(G_2)$ for some $q \in (2,+\infty]$ instead of being Lipschitz continuous.

To solve the Dirichlet problem in G_2, we first introduce two classes of Hardy spaces in G_2 (see Section 2). The first one, denoted by $H^p_\nu(G_2)$, is made of solutions of the conjugated Beltrami equation in G_2 satisfying a condition analogous to (3). The second one, denoted by $G^p_{A,B}(G_2)$, is made of so-called generalized analytic functions in G_2, also satisfying a condition analogous to (3). These two classes are related to each other by a trick going back to Bers and Nirenberg. Some properties of $G^p_{A,B}(G_2)$ are derived from the corresponding ones for the usual $H^p(G_2)$ space (made of analytic functions). We then solve the Dirichlet problem for generalized analytic functions in $G^p_{A,B}(G_2)$ and deduce the solution of the Dirichlet problem in $H^p_\nu(G_2)$.
We present the two classes of Hardy spaces in Section 2. Section 3 is devoted to the statement of the solution of the Dirichlet problem for generalized analytic functions, while Section 4 contains

the analogous statement for the conjugated Beltrami equation. We then prove the essential properties of $G^p_{A,B}(G_2)$ in Section 5. In Section 6, the results stated in Section 3 are established, and the solution of the Dirichlet problem for the conjugated Beltrami equation is derived in Section 7.

Remark 1.1. *We especially emphasize that the parametrization used in the present work for holomorphic functions in G_2 by the real part on one boundary and by the imaginary part on the other is a very explicit representation and is only valid for G_2. To extend the main results of this paper to higher multiplicities (i.e. multiply connected domains), it is possible to use other parametrizations of holomorphic functions in q-connected domains by potentials (see [9, 11]). This will be done in a forthcoming paper.*

Remark 1.2. *During the preparation of this manuscript, we learnt that L. Baratchart, Y. Fischer and J. Leblond ([3, 10]) considered generalized Hardy spaces on an annulus, in order to solve the Dirichlet problem for the equation $div(\sigma\nabla u) = 0$ with L^p boundary data, establish density results and solve bounded extremal problems in the spirit of [4]. Even if the generalized Hardy spaces are the same in the two works, the results obtained in the present work and in [3, 10] are of different nature.*

2 Two classes of Hardy spaces in the ring

2.1 Classical Hardy spaces

Let us first recall what the classical Hardy spaces on $\mathbb{D}$ and G_2 are ([7], Chapter 2 for $\mathbb{D}$ and Chapter 10 for G_2). Let $p \in [1, +\infty)$. Denote by $H^p(\mathbb{D})$ the space of holomorphic functions w in $\mathbb{D}$ such that

$$\|w\|_{H^p(\mathbb{D})} := \sup_{0<r<1} \|w\|_{L^p(\mathbb{T}_r)} < +\infty.$$

An essential feature of this space is that any function $w \in H^p(\mathbb{D})$ has a non tangential limit almost everywhere in $\mathbb{T}_1$, denoted by tr w, which belongs to $L^p(\mathbb{T}_1)$. One has

$$\|w\|_{H^p(\mathbb{D})} = \|\text{tr } w\|_{L^p(\mathbb{T}_1)}.$$

Moreover,

$$\lim_{r\to 1} \int_0^{2\pi} \left|w(re^{i\theta}) - \text{tr } w(e^{i\theta})\right|^p d\theta = 0.$$

A function $w : G_2 \to \mathbb{C}$ is said to belong to $H^p(G_2)$ if and only if w is holomorphic in G_2 and

$$\|w\|_{H^p(G_2)} := \sup_{r_0<r<1} \|w\|_{L^p(\mathbb{T}_r)} < +\infty.$$

Again, any function $w \in H^p(G_2)$ has a non tangential limit almost everywhere in ∂G_2, denoted by tr w. This non tangential limit belongs to $L^p(\partial G_2)$ and

$$\|\text{tr } w\|_{L^p(\partial G_2)} \sim \|w\|_{H^p(G_2)}. \tag{4}$$

Again, one has

$$\lim_{r\to r_0} \int_0^{2\pi} \left|w(re^{i\theta}) - \text{tr } w(r_0e^{i\theta})\right|^p d\theta = 0 \text{ and } \lim_{r\to 1} \int_0^{2\pi} \left|w(re^{i\theta}) - \text{tr } w(e^{i\theta})\right|^p d\theta = 0.$$

Let us also recall a classical topological decomposition of $H^p(G_2)$. Denote by $H^p(\mathbb{C}\setminus r_0\overline{\mathbb{D}})$ the space of holomorphic functions w in $\mathbb{C}\setminus r_0\overline{\mathbb{D}}$ such that

$$\|w\|_{H^p(\mathbb{C}\setminus r_0\overline{\mathbb{D}})} := \sup_{r>r_0} \|w\|_{L^p(\mathbb{T}_r)} < +\infty\,.$$

Any function in $H^p(\mathbb{C}\setminus r_0\overline{\mathbb{D}})$ has a trace on $\mathbb{T}_{r_0}$, which belongs to $L^p(\mathbb{T}_{r_0})$, and one defines $H^{p,0}(\mathbb{C}\setminus r_0\overline{\mathbb{D}})$ as the space of functions $w\in H^p(\mathbb{C}\setminus r_0\overline{\mathbb{D}})$ such that

$$\int_0^{2\pi} \mathrm{tr}\ w(r_0e^{i\theta})d\theta = 0.$$

Then, one has

$$H^p(G_2) = H^p(\mathbb{D})|_{G_2} \oplus H^p(\mathbb{C}\setminus r_0\overline{\mathbb{D}})|_{G_2} \tag{5}$$

and the decomposition is topological.

Finally, we recall a generalized Hilbert transform for the ring, already obtained in [8] under slightly stronger regularity assumptions:

Proposition 2.1.1. *Let $(u_1,v_2)\in L^p_{\mathbb{R}}(\mathbb{T}_{r_0})\times L^p_{\mathbb{R}}(\mathbb{T}_1)$. There exists a unique function $g\in H^p(G_2)$ such that*

$$\begin{cases} \mathit{Re}\ \mathit{tr}\ g = u_1 & \text{on } \mathbb{T}_{r_0}, \\ \mathit{Im}\ \mathit{tr}\ g = v_2 & \text{on } \mathbb{T}_1. \end{cases} \tag{6}$$

Moreover,

$$\|g\|_{H^p(G_2)} \le C_p\left(\|u_1\|_{L^p(\mathbb{T}_{r_0})} + \|v_2\|_{L^p(\mathbb{T}_1)}\right). \tag{7}$$

The operator

$$S(u_1,v_2) := \left(\mathit{Im}\ \mathit{tr}\ g|_{\mathbb{T}_{r_0}}, \mathit{Re}\ \mathit{tr}\ g|_{\mathbb{T}_1}\right)$$

is $L^p_{\mathbb{R}}(\mathbb{T}_{r_0})\times L^p_{\mathbb{R}}(\mathbb{T}_1)$-bounded.

As a corollary, one has:

Proposition 2.1.2. *Let $g\in H^p(G_2)$. Assume that*

$$\begin{cases} \mathit{Re}\ \mathit{tr}\ g = 0 & \text{on } \mathbb{T}_{r_0}, \\ \mathit{Im}\ \mathit{tr}\ g = 0 & \text{on } \mathbb{T}_1. \end{cases}$$

Then $g=0$ in G_2.

Propositions 2.1.1 and 2.1.2 will be proved in Appendix B.

2.2 New classes of Hardy spaces on G_2

Let us now introduce two classes of Hardy spaces on G_2, both generalizing $H^p(G_2)$. Let $q\in(2,+\infty)$ and $\nu\in W^{1,q}_{\mathbb{R}}(G_2)$. Note that $\nu\in L^\infty(G_2)$ by the Sobolev embeddings, and we always assume in the sequel that

$$\|\nu\|_\infty < 1 \tag{8}$$

and that

$$p > \frac{q}{q-2}. \tag{9}$$

Let $H^p_\nu(G_2)$ denote the space of measurable functions $f: G_2\to\mathbb{C}$ solving

$$\overline{\partial} f = \nu\overline{\partial f} \text{ in } G_2 \tag{10}$$

in the sense of distributions and satisfying furthermore

$$\underset{r_0<r<1}{\text{ess sup}} \|f\|_{L^p(\mathbb{T}_r)} < +\infty. \tag{11}$$

Equipped with the norm

$$\|f\|_{H^p_\nu(G_2)} := \underset{r_0<r<1}{\text{ess sup}} \|f\|_{L^p(\mathbb{T}_r)}, \tag{12}$$

$H^p_\nu(G_2)$ is a Banach space. Clearly, when $\nu = 0$, $H^p_\nu(G_2)$ coincides with the classical $H^p(G_2)$ space.

The second class of Hardy spaces we consider is made of generalized analytic functions in G_2 (see [14]). Let p and q as before and $A, B \in L^q(G_2)$. By "generalized analytic functions", we mean solutions of

$$\overline{\partial} w = Aw + B\overline{w} \text{ in } G_2 \tag{13}$$

in the sense of distributions. Denote by $G^p_{A,B}(G_2)$ the space of all measurable functions w on G_2 solving equation (13) in the sense of distributions and satisfying

$$\underset{r_0<r<1}{\text{ess sup}} \|w\|_{L^p(\mathbb{T}_r)} < +\infty, \tag{14}$$

equipped with the norm

$$\|w\|_{G^p_{A,B}(G_2)} := \underset{r_0<r<1}{\text{ess sup}} \|w\|_{L^p(\mathbb{T}_r)}. \tag{15}$$

It is also a Banach space, which is obviously equal to $H^p(G_2)$ when $A = B = 0$.

Let us now summarize essential properties of these spaces. We begin with $G^p_{A,B}(G_2)$:

Proposition 2.2.1. 1. *For any $w \in G^p_{A,B}(G_2)$, there exist $\widetilde{w} \in C^\alpha(\overline{G_2})$ for all $\alpha \in \left(0, 1 - \frac{2}{q}\right)$ and $F \in H^p(G_2)$ such that $w = e^{\widetilde{w}} F$. One has $\|\widetilde{w}\|_\infty \leq C$ where $C > 0$ only depends on A and B. Moreover, $\widetilde{w}$ can be chosen in such a way that Im $\widetilde{w} = 0$ on ∂G_2.*

2. *Any function $w \in G^p_{A,B}(G_2)$ has a non tangential limit at almost every point $\xi \in \partial G_2$, denoted by tr $w(\xi)$. Moreover, tr $w \in L^p(\partial G_2)$ and, for all $w \in G^p_{A,B}(G_2)$,*

$$\|tr\ w\|_{L^p(\partial G_2)} \sim \|w\|_{G^p_{A,B}(G_2)}.$$

Finally, for all $w \in G^p_{A,B}(G_2)$,

$$\lim_{r\to r_0} \int_0^{2\pi} \left| w(re^{i\theta}) - tr\ w(r_0 e^{i\theta}) \right|^p d\theta = 0 \text{ and } \lim_{r\to 1} \int_0^{2\pi} \left| w(re^{i\theta}) - tr\ w(e^{i\theta}) \right|^p d\theta = 0. \tag{16}$$

3. *Any function $w \in G^p_{A,B}(G_2)$ belongs to $L^{p_1}(G_2)$ for all $p_1 \in [p, 2p)$ and*

$$\|w\|_{L^{p_1}(G_2)} \leq C_{p_1} \|w\|_{G^p_{A,B}(G_2)}.$$

4. *If $w \in G^p_{A,B}(G_2)$, Re tr $w = 0$ on $\partial\mathbb{T}_{r_0}$ and Im tr $w = 0$ on $\partial\mathbb{T}_1$, then $w = 0$.*

Note that the principle of the factorization given by assertion 1. actually goes back to Bers and Vekua (see [14], see also [5, 6]). The proof of this proposition will be given in Section 5.
The link between H^p_ν and $G^p_{A,B}$ is given by a trick which originally appeared in [6]. Given $\nu \in W^{1,q}_{\mathbb{R}}(G_2)$ satisfying (8), define

$$B = \frac{\overline{\partial}\nu}{\sqrt{1-\nu^2}} \in L^q(G_2).$$

Then $f \in H^p_\nu(G_2)$ if and only if the function w defined by

$$w := \frac{f - \nu\overline{f}}{\sqrt{1-\nu^2}} = \sqrt{\frac{1-\nu}{1+\nu}}\mathrm{Re}\ f + i\sqrt{\frac{1+\nu}{1-\nu}}\mathrm{Im}\ f \tag{17}$$

belongs to $G^p_{0,B}(G_2)$ (see [4]). Using the fact that (17) is equivalent to $f = \frac{w+\nu\overline{w}}{\sqrt{1-\nu^2}}$ and that ν is continuous in $\overline{G_2}$ by the Sobolev embeddings, we derive from Proposition 2.2.1 the following properties of $H^p_\nu(G_2)$:

Proposition 2.2.2. 1. *Any function $f \in H^p_\nu(G_2)$ has a non tangential limit at almost every point $\xi \in \partial G_2$, denoted by $tr\ f(\xi)$. Moreover, $tr\ f \in L^p(\partial G_2)$ and, for all $f \in H^p_\nu(G_2)$,*

$$\|tr\ f\|_{L^p(\partial G_2)} \sim \|f\|_{H^p_\nu(G_2)}.$$

Finally, for all $f \in H^p_\nu(G_2)$,

$$\lim_{r\to r_0}\int_0^{2\pi}\left|f(re^{i\theta}) - tr\ f(r_0e^{i\theta})\right|^p d\theta = 0 \ \textit{and}\ \lim_{r\to 1}\int_0^{2\pi}\left|f(re^{i\theta}) - tr\ f(e^{i\theta})\right|^p d\theta = 0. \tag{18}$$

2. *If $f \in H^p_\nu(G_2)$, Re $tr\ f = 0$ a.e. on $\mathbb{T}_{r_0}$ and Im $tr\ f = 0$ a.e. on $\mathbb{T}_1$, then $f = 0$ in G_2.*

Remark 2.1. *If, instead of* (17)*, we define*

$$w = f - \nu\overline{f},$$

then a straightforward computation yields that $f \in H^p_\nu(G_2)$ if and only if $w \in G^p_{A,B}(G_2)$ with

$$A = -\frac{\nu\overline{\partial}\nu}{1-\nu^2},\ B = -\frac{\overline{\partial}\nu}{1-\nu^2}.$$

3 The Dirichlet problem for generalized analytic functions in the ring

As in [4], Theorem 4.4.1.2, we solve the Dirichlet problem associated to equation (13) in $G^p_{A,B}(G_2)$. More precisely:

Theorem 3.1. *Let $p \in (1,+\infty)$. For all $\overrightarrow{\varphi} = (\varphi_1,\varphi_2) \in L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, there exists a unique function $w \in G^p_{A,B}(G_2)$ such that*

$$\begin{cases} Re\ tr\ w = \varphi_1 & a.e.\ on\ \mathbb{T}_{r_0}, \\ Im\ tr\ w = \varphi_2 & a.e.\ on\ \mathbb{T}_1. \end{cases} \tag{19}$$

Moreover, there exists $C_{p,A,B,r_0} > 0$ only depending on p, A, B and r_0 such that

$$\|w\|_{G^p_{A,B}(G_2)} \le C_{p,A,B,r_0}\left(\|\varphi_1\|_{L^p(\mathbb{T}_{r_0})} + \|\varphi_2\|_{L^p(\mathbb{T}_1)}\right). \tag{20}$$

Remark 3.1. 1. *Note the form of the boundary condition* (19)*: we prescribe the real part of w on the inner circle and its imaginary part on the outer circle. Even when $A = B = 0$, i.e. for holomorphic functions, it is not possible in general to prescribe the real part of w on both circles. Indeed, let $u_1 \in L^2(\mathbb{T}_{r_0})$ and $u_2 \in L^2(\mathbb{T}_1)$ be real-valued and assume that there exists a holomorphic function w in G_2 such that*

$$Re\ w = u_1\ on\ \mathbb{T}_{r_0}\ and\ Re\ w = u_2\ on\ \mathbb{T}_1.$$

Writing $u_1(r_0e^{it}) = \sum_{n\in\mathbb{Z}} u_{1,n}r_0^n e^{int}$, $u_2(e^{it}) = \sum_{n\in\mathbb{Z}} u_{2,n}e^{int}$ *and* $w(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, *computations analogous to [8], p. 948, yield*

$$u_{1,n} = a_n r_0^n + \overline{a_{-n}} r_0^{-n}$$

and

$$u_{2,n} = a_n + \overline{a_{-n}}$$

for all $n \in \mathbb{Z}$*. In particular,* $u_{1,0} = u_{2,0}$*. For more on this, see [11].*

2. *Let us point out a difference with Theorem 4.4.1.2 of [4]: in the disk, if the real part of* w *is prescribed on the boundary, then the solution of the Dirichlet problem in the corresponding Hardy space is unique up to an imaginary constant. Here, once the real part of* w *on the inner circle and the imaginary part on the outer one are fixed, the solution is unique.*

Theorem 3.1 will be established in Section 6.

4 The Dirichlet problem for the conjugated Beltrami equation in the ring

We conclude with the solution of the Dirichlet problem in $H^p_\nu(G_2)$:

Theorem 4.1. *For all* $\overrightarrow{\varphi} = (\varphi_1, \varphi_2) \in L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$*, there uniquely exists* $f \in H^p_\nu(G_2)$ *such that:*

$$\begin{cases} Re\ tr\ f = \varphi_1 & a.e.\ on\ \mathbb{T}_{r_0}, \\ Im\ tr\ f = \varphi_2 & a.e.\ on\ \mathbb{T}_1. \end{cases} \tag{21}$$

Moreover, there exists $C_{p,\nu,r_0} > 0$ *only depending on* p, ν *and* r_0 *such that:*

$$\|f\|_{H^p_\nu(G_2)} \le C_{p,\nu,r_0} \left(\|\varphi_1\|_{L^p(\mathbb{T}_{r_0})} + \|\varphi_2\|_{\mathbb{T}_1} \right). \tag{22}$$

5 Proofs of the properties of Hardy spaces

This section is devoted to the proof of Proposition 2.2.1. Assertion 1. is a slightly modified version of the similarity principle stated in [9], Theorem 2.1, in the more general context of multiply connected domains, under the extra assumption that $w \in C^\beta(\overline{G_2})$ for some $\beta \in (0,1)$. We provide here a quick proof for the reader's convenience.
Let $e : G_2 \to \mathbb{R}$ be the solution of

$$\begin{cases} \Delta e = 0 & \text{in } G_2, \\ e = 0 & \text{on } \mathbb{T}_1, \\ e = 1 & \text{on } \mathbb{T}_{r_0}. \end{cases}$$

Set

$$a := \int_{\mathbb{T}_{r_0}} \frac{\partial e}{\partial n} d\sigma,$$

where $\frac{\partial}{\partial n}$ stands for the normal derivative and $d\sigma$ for the surface measure on ∂G_2. By the Hopf lemma, $a > 0$. Define

$$c := a^{-1} > 0.$$

Consider the function ψ defined on ∂G_2 by

$$\psi(z) = 0 \text{ if } z \in \mathbb{T}_1,\ \psi(z) = \alpha \text{ if } z \in \mathbb{T}_{r_0}, \tag{23}$$

where $\alpha \in \mathbb{R}$ will be chosen later. Define also, for all $z \in G_2$,

$$g(z) = \begin{cases} A(z) + B(z)\dfrac{\overline{w(z)}}{w(z)} & \text{if } w(z) \neq 0, \\ 0 & \text{if } w(z) = 0. \end{cases}$$

Applying Theorem 4.5 in [9] with the function ψ given by (23) yields a function $\widetilde{w} \in C^{0,\gamma}(\overline{G_2})$ for some $\gamma \leq 1 - \frac{2}{q}$ (this follows from [14] and holds whenever w is measurable) such that $w = e^{\widetilde{w}}F$ where F is holomorphic in G_2,

$$\mathrm{Im}\ \widetilde{w} = 0 \text{ on } \mathbb{T}_1$$

and

$$\begin{aligned} \mathrm{Im}\ \widetilde{w} &= \alpha + c\alpha \int_{\mathbb{T}_{r_0}} \frac{\partial e}{\partial n} d\sigma - 4\mathrm{Im} \iint_{G_2} g(\zeta)\partial e(\zeta) d\zeta \wedge d\overline{\zeta} \\ &= 2\alpha - 4\mathrm{Im} \iint_{G_2} g(\zeta)\partial e(\zeta) d\zeta \wedge d\overline{\zeta} \text{ on } \mathbb{T}_{r_0}. \end{aligned}$$

Choosing α appropriately therefore gives $\mathrm{Im}\ \widetilde{w} = 0$ on ∂G_2. Finally, since w satisfies (14) and $\widetilde{w}$ is bounded in G_2 by a constant only depending on A and B, F also satisfies (14). □

Assertion 2. follows at once from assertion 1. and the fact that $\widetilde{w}$ is continuous in $\overline{G_2}$. For assertion 3, in view of assertion 1, it is clearly enough to establish the conclusion for functions in $H^p(G_2)$. But this follows from (5) and the fact that the corresponding property holds for functions in $H^p(\mathbb{D})$ (Lemma 5.2.1 in [4]) and therefore also for functions in $H^p(\mathbb{C} \setminus r_0\overline{\mathbb{D}})$, since

$$w \in H^p(\mathbb{C} \setminus r_0\overline{\mathbb{D}}) \Leftrightarrow z \mapsto \overline{w\left(\frac{r_0}{\overline{z}}\right)} \in H^p(\mathbb{D}).$$

Finally, let $w \in G^p_{A,B}(G_2)$ satisfy the assumptions of assertion 4. Write $w = e^{\widetilde{w}}F$ as in assertion 1. Since $\widetilde{w}$ is real-valued on ∂G_2, an easy computation shows that F satisfies the assumptions of Proposition 2.1.2. As a consequence, $F = 0$ and $w = 0$. □

6 Solving the Dirichlet problem for generalized analytic functions

The proof is divided in two steps: we first solve a different Dirichlet type problem, prescribing the analytic projection of the trace of the solution, from which we derive the conclusion of Theorem 3.1.

6.1 The analytic projection

We consider here a version of the analytic projection adapted to the case of the ring (see [7]). Given $\overrightarrow{\varphi} = (\varphi_1, \varphi_2) \in L^p(\mathbb{T}_{r_0}) \times L^p(\mathbb{T}_1)$, define, for all $z \in G_2$,

$$\mathcal{C}(\overrightarrow{\varphi})(z) := \frac{1}{2\pi} \int_{\mathbb{T}_{r_0}} \frac{\varphi_1(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi} \int_{\mathbb{T}_1} \frac{\varphi_2(\zeta)}{\zeta - z} d\zeta,$$

where, in the first integral, $\mathbb{T}_{r_0}$ is described clockwise and $\mathbb{T}_1$ is described counterclockwise. The function $\mathcal{C}(\overrightarrow{\varphi})$ is holomorphic in G_2 and actually belongs to the Hardy space $H^p(G_2)$. It therefore has a non tangential limit at almost every point of ∂G_2, and we set

$$P_+(\overrightarrow{\varphi}) := \left(\mathrm{tr}\ \mathcal{C}(\overrightarrow{\varphi})|_{\mathbb{T}_{r_0}}, \mathrm{tr}\ \mathcal{C}(\overrightarrow{\varphi})|_{\mathbb{T}_1}\right).$$

Note that P_+ is $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$-bounded.

6.2 The Dirichlet problem for generalized analytic functions with prescribed analytic projection

Our first step towards Theorem 3.1 is the solution of the Dirichlet problem for generalized analytic functions with prescribed analytic projection:

Theorem 6.2.1. *Let $p \in (1, +\infty)$. For all $g \in H^p(G_2)$, there exists a unique $w \in G^p_{A,B}(G_2)$ such that*

$$P_+ \left(tr\ w\right) = \left(tr\ g|_{\mathbb{T}_{r_0}}, tr\ g|_{\mathbb{T}_1}\right). \tag{24}$$

Moreover,

$$\|w\|_{G^p_{A,B}(G_2)} \leq C_p \|g\|_{H^p(G_2)}. \tag{25}$$

Proof: the argument is inspired by the one of Theorem 4.4.1.1 in [4]. Consider the operator T defined, for all $w \in L^p(G_2)$ and all $z \in G_2$ by

$$Tw(z) := \iint_{G_2} \frac{w(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

Define also, for all $f \in L^p(\mathbb{C})$ and all $z \in \mathbb{C}$,

$$\check{T}f(z) := \iint_{G_2} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

We claim:

Proposition 6.2.1.

1. *The operator T is bounded from $L^p(G_2)$ to $W^{1,p}(G_2)$ and compact on $L^p(G_2)$. Moreover, for all $w \in L^p(G_2)$,*

$$\overline{\partial}(Tw) = w. \tag{26}$$

2. *The operator $\check{T}$ is bounded from $L^p(\mathbb{C})$ to $W^{1,p}_{loc}(\mathbb{C})$.*

3. *Let $w \in L^p(G_2)$ and $g \in H^p(G_2)$. Assume that*

$$w = g + T(Aw + B\overline{w}).$$

Then there exists $p_0 > 2$ such that $Aw + B\overline{w} \in L^{p_0}(G_2)$ and

$$\|Aw + B\overline{w}\|_{L^{p_0}(G_2)} \leq C \|g\|_{H^p(G_2)}. \tag{27}$$

4. *The operator $w \mapsto w - T\left(Aw + B\overline{w}\right)$ is an isomorphism from $L^p(G_2)$ onto itself.*

5. *For all $w \in G^p_{A,B}(G_2)$,*

$$w = \mathcal{C}(tr\ w) + T(Aw + B\overline{w}), \text{ a.e. in } G_2. \tag{28}$$

6. *If $w \in G^p_{A,B}(G_2)$ and $P_+ \left(tr\ w\right) = 0$ a.e. on ∂G_2, then $w(z) = 0$ for all $z \in G_2$.*

The proof of this proposition will be given in Appendix A. Relying on the conclusions of Proposition 6.2.1, let us conclude the proof of Theorem 6.2.1. Proposition 6.2.1, assertion 4, yields a function $w \in L^p(G_2)$ such that

$$w = g + T\left(Aw + B\overline{w}\right).$$

Since g is holomorphic in G_2, assertion 1. in Proposition 6.2.1 shows that $\overline{\partial} w = Aw + B\overline{w}$. Moreover, since $g \in H^p(G_2)$, it follows from item 3 in Proposition 6.2.1 that $Aw + B\overline{w} \in L^{p_0}$ for some $p_0 > 2$ with estimate (27), and therefore $T(Aw + B\overline{w}) \in W^{1,p_0}(G_2) \subset L^\infty(G_2)$, with

$$\|T(Aw + B\overline{w})\|_{L^\infty(G_2)} \leq C \|g\|_{H^p(G_2)}.$$

As a consequence, $w \in G^p_{A,B}(G_2)$ and (25) holds. Formula (28) now shows that $g = \mathcal{C}\left(\text{tr } w\right)$ and therefore $\left(\text{tr } g|_{\mathbb{T}_{r_0}}, \text{tr } g|_{\mathbb{T}_1}\right) = P_+ \left(\text{tr } w\right)$. Uniqueness of w follows from assertion 6. in Proposition 6.2.1. □

6.3 Solution of the Dirichlet problem for generalized analytic functions

Let us conclude the proof of Theorem 3.1, arguing as for the proof of Theorem 4.4.1.2 in [4]. Define $\mathcal{T} : G^p_{A,B}(G_2) \to L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$ by

$$\mathcal{T}w = \left(\text{Re tr } w|_{\mathbb{T}_{r_0}}, \text{Im tr } w|_{\mathbb{T}_1}\right).$$

The operator $\mathcal{T}$ is bounded from $G^p_{A,B}(G_2)$ to $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, and the conclusion of Theorem 3.1 exactly means that $\mathcal{T}$ is an isomorphism from $G^p_{A,B}(G_2)$ onto $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$.
In order to establish this fact, we define an operator $\mathcal{S}$ from $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$ to $G^p_{A,B}(G_2)$ in the following way. For all $\overrightarrow{\psi} = (\psi_1, \psi_2) \in L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, Proposition 2.1.1 yields the unique function $g \in H^p(G_2)$ such that

$$\begin{cases} \text{Re tr } g = \psi_1 & \text{on } \mathbb{T}_{r_0}, \\ \text{Im tr } g = \psi_2 & \text{on } \mathbb{T}_1, \end{cases}$$

with

$$\|g\|_{H^p(G_2)} \leq C \left(\|\psi_1\|_{L^p(\mathbb{T}_{r_0})} + \|\psi_2\|_{L^p(\mathbb{T}_1)} \right). \tag{29}$$

Define now $w := \mathcal{S}(\psi_1, \psi_2)$ as the unique function $w \in G^p_{A,B}(G_2)$ (given by Theorem 6.2.1) such that $P_+(\text{tr } w) = \left(\text{tr } g|_{\mathbb{T}_{r_0}}, \text{tr } g|_{\mathbb{T}_1}\right)$. Recall also that

$$\|w\|_{G^p_{A,B}(G_2)} \leq C \|g\|_{H^p(G_2)}. \tag{30}$$

Thus, (29) and (30) show that $\mathcal{S}$ is continuous. It is plain to see that $\mathcal{S}$ is one-to-one on $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$. Moreover, let $w \in G^p_{A,B}(G_2)$. If $g = \mathcal{C}(\text{tr } w)$, one has $g \in H^p(G_2)$ and $P_+(\text{tr } w) = \text{tr } g$. Setting $\varphi_1 = \text{Re tr } g|_{\mathbb{T}_{r_0}}$ and $\varphi_2 = \text{Im tr } g|_{\mathbb{T}_1}$, one has $\mathcal{S}(\varphi_1, \varphi_2) = w$, which shows that $\mathcal{S}$ is onto. Therefore, $\mathcal{S}$ is an isomorphism from $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$ onto $G^p_{A,B}(G_2)$.
To check that $\mathcal{T}$ is an isomorphism from $G^p_{A,B}(G_2)$ onto $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, it is therefore enough to check that $\mathcal{A} := \mathcal{T} \circ \mathcal{S}$ is an isomorphism from $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$ onto itself.
The operator $\mathcal{A}$ is $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$-bounded. Moreover, formula (28) yields that, for all $\overrightarrow{\psi} \in L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, one has

$$\mathcal{A}\overrightarrow{\psi} = \overrightarrow{\psi} + \mathcal{B}\overrightarrow{\psi}$$

where

$$\mathcal{B}\overrightarrow{\psi} := \left(\text{Re tr } \left(T\left(Aw + B\overline{w}\right)\right)|_{\mathbb{T}_{r_0}}, \text{Im tr } \left(T\left(Aw + B\overline{w}\right)\right)|_{\mathbb{T}_1}\right)$$

and $w := \mathcal{S}(\overrightarrow{\psi})$. If $g := \mathcal{C}(\text{tr } w)$, (28) shows that $w = g + T(Aw + B\overline{w})$ and item 3. in Proposition 6.2.1 therefore yields that $Aw + B\overline{w} \in L^{p_0}(G_2)$ for some $p_0 > 2$ and

$$\|Aw + B\overline{w}\|_{L^{p_0}(G_2)} \leq C \|g\|_{H^p(G_2)} \leq C \left(\|\psi_1\|_{L^p(\mathbb{T}_{r_0})} + \|\psi_2\|_{L^p(\mathbb{T}_1)} \right),$$

so that $T(Aw + B\overline{w}) \in W^{1,p_0}(G_2)$ and

$$\|T(Aw + B\overline{w})\|_{W^{1,p_0}(G_2)} \leq C \left(\|\psi_1\|_{L^p(\mathbb{T}_{r_0})} + \|\psi_2\|_{L^p(\mathbb{T}_1)} \right).$$

As a consequence, and since $W^{1,p_0}(G_2) \subset C^{0,\gamma}(\overline{G_2})$ with $\gamma := 1 - \frac{2}{p_0}$, the operator $\mathcal{B}$ is bounded from $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$ to $C^{0,\gamma}(\mathbb{T}_{r_0}) \times C^{0,\gamma}(\mathbb{T}_1)$, and is therefore compact on $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$. Since, by Proposition 2.2.1, assertion 4, $\mathcal{T}$, and therefore $\mathcal{A}$, are injective on $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, it follows that $\mathcal{A}$ is actually an isomorphism from $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$ onto itself. Thus, $\mathcal{T}$ is an isomorphism from $G^p_{A,B}(G_2)$ onto $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, which yields the existence and the uniqueness of w. Finally, (20) follows from the boundedness of $\mathcal{T}^{-1}$. □

7 Solution of the Dirichlet problem for the conjugated Beltrami equation

We establish now Theorem 4.1. Define

$$\sigma := \frac{1-\nu}{1+\nu},$$

and note that, because of (8), there exist $0 < c < C$ such that $c \leq \sigma(z) \leq C$ for almost every $z \in G_2$. Set $\psi_1 = \varphi_1 \sigma^{1/2} \in L^p_{\mathbb{R}}(\mathbb{T}_{r_0})$ and $\psi_2 = \varphi_2 \sigma^{-1/2} \in L^p_{\mathbb{R}}(\mathbb{T}_1)$. Theorem 3.1 yields the unique function $w \in G^p_{0,B}(G_2)$ such that

$$\begin{cases} \text{Re } (\text{tr } w) = \psi_1 & \text{a.e. on } \mathbb{T}_{r_0}, \\ \text{Im } (\text{tr } w) = \psi_2 & \text{a.e. on } \mathbb{T}_1. \end{cases}$$

If $f := \frac{w+\nu\overline{w}}{\sqrt{1-\nu^2}}$, then $f \in H^p_\nu(G_2)$ and, as in the proof of Theorem 4.4.2.1 in [4], satisfies (21) and (22). Uniqueness of f follows from Proposition 2.2.2, assertion 3.

A Appendix: Proof of the properties of some operators

Proof of Proposition 6.2.1: the proofs of assertions 1. and 2. are identical to the corresponding ones in the case of the disk (see assertion 4. in Proposition 5.2.1 in [4]).

Let us now turn to point 3. We first check that $Aw + B\overline{w} \in L^{p_0}(G_2)$ for some $p_0 > 2$. The Hölder inequality yields that $Aw + B\overline{w} \in L^r(G_2)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.
Assume first that $p > \frac{2q}{q-2}$. In that case, $r > 2$, and we are done.
Assume now that $p = \frac{2q}{q-2}$, so that $r = 2$. Then $T(Aw + B\overline{w}) \in W^{1,2}(G_2) \subset L^t(G_2)$ for all $t < +\infty$. As a consequence, since $g \in L^s(G_2)$ for all $s \in (1, 2p)$ (Proposition 2.2.1, item 3.), $w \in L^s(G_2)$ for all $s \in (1, 2p)$. Since $\lim_{s \to 2p} \frac{1}{q} + \frac{1}{s} = \frac{1}{q} + \frac{1}{2p} = \frac{1}{r} - \frac{1}{2p} < \frac{1}{2}$, there exists $s \in (1, 2p)$ such that $\frac{1}{p_0} := \frac{1}{q} + \frac{1}{s} < \frac{1}{2}$. Thus, $Aw + B\overline{w} \in L^{p_0}(G_2)$.
Assume finally that $p < \frac{2q}{q-2}$, so that $r < 2$. Then $T(Aw + B\overline{w}) \in W^{1,r}(G_2) \subset L^{r^*}(G_2)$ with $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{2}$. Since furthermore $p > \frac{q}{q-2}$ by assumption (9), one has $r^* > 2p$, so that again $w \in L^s(G_2)$ for all $s \in (1, 2p)$. Therefore, for all $s \in (1, 2p)$, if $\frac{1}{p_0} = \frac{1}{q} + \frac{1}{s}$, one has $Aw + B\overline{w} \in L^{p_0}(G_2)$. Since $\frac{1}{q} + \frac{1}{2p} = \frac{1}{r^*} - \frac{1}{2p} + \frac{1}{2} < \frac{1}{2}$, one concludes as before.

We will now establish (27) and assertion 4. simultaneously, making use of the following notation: for any function u on G_2, denote by $\breve{u}$ its extension by 0 outside G_2.
Define $T_1(w) := T(Aw + B\overline{w})$ for $w \in L^p(G_2)$, and observe first that T_1 is compact on $L^p(G_2)$. Indeed, since $A, B \in L^q(G_2)$ and $w \in L^p(G_2)$, $Aw + B\overline{w} \in L^r(G_2)$ with $r = \frac{pq}{p+q}$. It follows from assertion 1 that T_1 is bounded from $L^p(G_2)$ to $W^{1,r}(G_2)$, and this space is always compactly embedded in $L^p(G_2)$. Indeed, this is immediate when $r \geq 2$, and if $r < 2$, this follows from the fact that $p < r^* := \frac{2r}{2-r}$ since $q > 2$.
To prove that $I - T_1$ is an isomorphism from $L^p(G_2)$ onto itself, it is therefore enough to check that it is one to one. Let $w \in L^p(G_2)$ such that $w = T_1 w = T(Aw + B\overline{w})$. Assertion 3 shows that $Aw + B\overline{w} \in L^{p_0}(G_2)$ for some $p_0 > 2$. Set now $u = \breve{T}(\overbrace{Aw + B\overline{w}}) \in W^{1,p_0}_{loc}(\mathbb{C})$.
It holds in the sense of distributions that

$$\bar{\partial} u = \overbrace{Aw + B\overline{w}} = \breve{A} u + \breve{B}\, \overline{u} \quad \text{a.e. in } \mathbb{C}. \tag{31}$$

In addition, $u(z)$ clearly goes to 0 when $|z|$ goes to $+\infty$. It now follows from the generalized Liouville theorem [2, Prop. 3.3] that $u = 0$, therefore $w = 0$.
Coming back to assertion 3., if $w = g + T(Aw + B\overline{w})$, with $w \in L^p(G_2)$ and $g \in H^p(G_2) \subset L^p(G_2)$, one deduces from assertion 4. that $w = (I - T_1)^{-1} g$, which yields

$$\|w\|_{L^p(G_2)} \le C \|g\|_{L^p(G_2)}.$$

Estimate (27) follows. Indeed, when $p > \frac{2q}{q-2} > 2$,

$$\|Aw + B\overline{w}\|_{L^r(G_2)} \le C \|w\|_{L^p(G_2)} \le C \|g\|_{L^p(G_2)} \le C \|g\|_{H^p(G_2)},$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. When $p = \frac{2q}{q-2}$, one has, for all $t < +\infty$,

$$\|T(Aw + B\overline{w})\|_{L^t(G_2)} \le C \|T(Aw + B\overline{w})\|_{W^{1,2}(G_2)} \le C \|Aw + B\overline{w}\|_{L^2(G_2)} \le C \|w\|_{L^p(G_2)} \le C \|g\|_{L^p(G_2)},$$

and since

$$\|g\|_{L^s(G_2)} \le C \|g\|_{H^p(G_2)}$$

for all $s \in (1, 2p)$, (27) follows. Finally, when $p < \frac{2q}{q-2}$,

$$\|T(Aw + B\overline{w})\|_{L^{r^*}(G_2)} \le C \|T(Aw + B\overline{w})\|_{W^{1,r}(G_2)} \le C \|Aw + B\overline{w}\|_{L^r(G_2)} \le C \|w\|_{L^p(G_2)},$$

and one concludes similarly.

For assertion 5., consider now $w \in G^p_{A,B}(G_2)$. By assertion 1., $\overline{\partial}\left(w - T(Aw + B\overline{w})\right) = 0$ in the sense of distributions, so that the function $w - T(Aw + B\overline{w})$ is holomorphic in G_2, and therefore belong to $W^{1,r}_{loc}(G_2)$ for all $r \in (1, +\infty)$. Since $T(Aw + B\overline{w}) \in W^{1,r}(G_2)$, we obtain $w \in W^{1,r}_{loc}(G_2)$ for all $r \in (1, +\infty)$. For all $\varepsilon > 0$, the Cauchy-Green formula therefore yields

$$w(z) = \frac{1}{2\pi i} \int_{\mathbb{T}_{r_0+\varepsilon}} \frac{w(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\mathbb{T}_{1-\varepsilon}} \frac{w(\zeta)}{\zeta - z} d\zeta + T\left((Aw + B\overline{w}) \chi_{G_{2,\varepsilon}} \right)(z), \quad r_0 + \varepsilon < |z| < 1 - \varepsilon, \tag{32}$$

with

$$G_{2,\varepsilon} := \{ z \in \mathbb{C};\ r_0 + \varepsilon < |z| < 1 - \varepsilon \}.$$

Letting $\varepsilon \to 0$ in (32), and using (16) for the two first terms and dominated convergence and assertion 1. for the third one, we obtain (28).
Finally, for point 6., assume that $w \in H^p(G_2)$ and $P_+(\text{tr } w) = 0$ a.e. on ∂G_2. The function $\mathcal{C}(\text{tr w})$ is in $H^p(G_2)$ and its trace vanishes on ∂G_2, which entails that it is zero in G_2. Formula (28) therefore yields that $w = T(Aw + B\overline{w})$, which in turn, by assertion 4., shows that $w = 0$.

B Appendix: Proof of some properties of functions in $H^p(G_2)$

Proof of Proposition 2.1.1: we argue similarly as in [8], Theorem 2.2. For all $k \in \mathbb{Z}$, define

$$u_{1,k} := \frac{1}{2\pi} \int_0^{2\pi} u_1(r_0 e^{i\theta}) e^{-ik\theta} d\theta \text{ and } v_{2,k} := \frac{1}{2\pi} \int_0^{2\pi} v_2(e^{i\theta}) e^{-ik\theta} d\theta.$$

The proof of Theorem 2.2 in [8] shows that, if a function g satisfying the conclusions of Proposition 2.1.1 exists, then one has $g(z) = \sum_{k \in \mathbb{Z}} a_k z^k$ in G_2, with

$$a_k := 2 \frac{r_0^k u_{1,k} + i v_{2,k}}{r_0^{2k} + 1}. \tag{33}$$

This already proves uniqueness of g.
Recall now that, according to Theorem 2.3 in [8], for all functions $f_1 \in L^2_{\mathbb{R}}(\mathbb{T}_{r_0})$ and $g_2 \in L^2_{\mathbb{R}}(\mathbb{T}_1)$, there exists a unique holomorphic function w in G_2 such that Re $w = f_1$ on $\mathbb{T}_{r_0}$ and Im $w = g_2$ on $\mathbb{T}_1$. If the operator S is defined by $w = S(f_1, g_2)$, Theorem 2.5 in [8] shows that S can be written as

$$S(f_1, g_2) = \left(\mathcal{H}_0 f_1 + \widehat{A} f_1 + \widehat{B} g_2, \mathcal{H}_0 g_2 + \widehat{C} f_1 + \widehat{D} g_2\right)$$

where $\mathcal{H}_0$ stands for the usual Hilbert transform and $\widehat{A}, \widehat{B}, \widehat{C}$ and $\widehat{D}$ are linear integral operators with analytic kernels. This shows that S extends to an $L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$-bounded operator.
Given now $u_1, v_2 \in L^p_{\mathbb{R}}(\mathbb{T}_{r_0}) \times L^p_{\mathbb{R}}(\mathbb{T}_1)$, set $(u_2, v_1) = S(u_1, v_2)$ and

$$\overrightarrow{\psi} := (u_1 + iu_2, v_1 + iv_2).$$

Define now

$$g := \mathcal{C}\left(\overrightarrow{\psi}\right).$$

Since $\overrightarrow{\psi} \in L^p(\mathbb{T}_{r_0}) \times L^p(\mathbb{T}_1)$, the function g belongs to $H^p(G_2)$ and the definition of $\overrightarrow{\psi}$ yields that (6) and (7) hold. □

Proof of Proposition 2.1.2: it is an immediate corollary of Proposition 2.1.1. □

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